Couple Curves Agregation

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Abstract

Do you prefer to win 10 or to play heads or tails 0 or 20 ? [8] Most people do not like risk but in modifying the amount and asking several questions a concave curve called "Utility function" [3] can represent your investor prole. Utility curves are behavioral economics [10] basic. Given your Utility curve and your spouse's one, a more complex issue is :

"What will be the Utility profile of your household ?"

To try to answer this question we need first to draw your and your spouse prole, assuming we can do it which mathematical operation enables to aggregate two utility profile ? A sum ? A multiplication ? A convolution ? None of those, in fact, we will see it could be a kind of mean but it's not so simple !

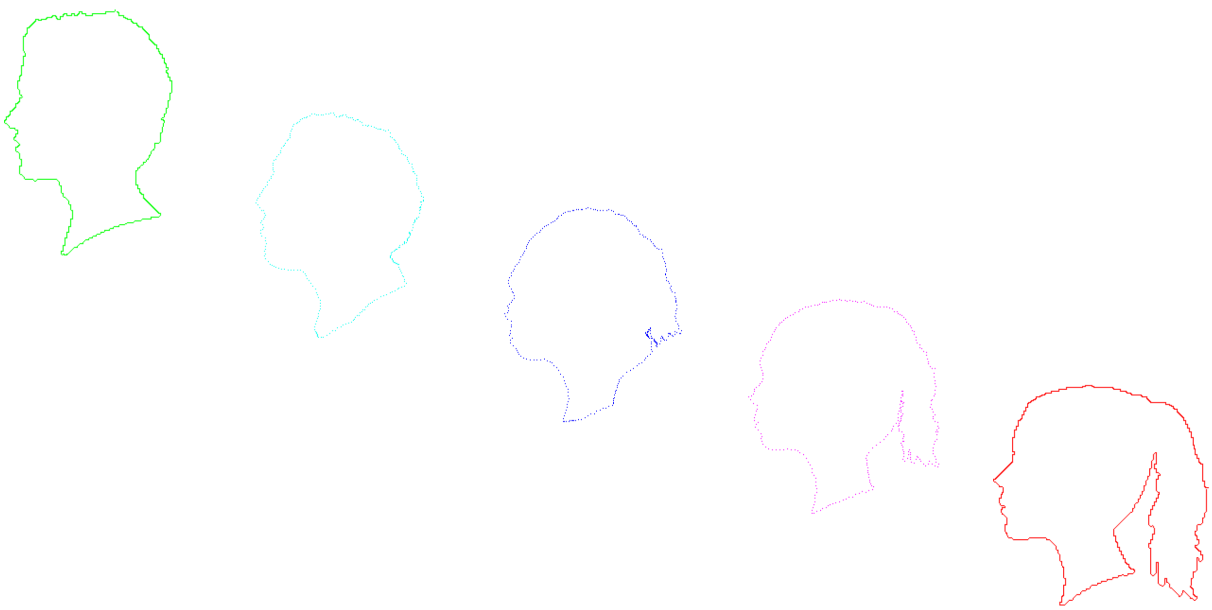


FIGURE 1 – Visage

Introduction : Behavioral Economics

"We are not Econs" this little sentence from Richard H.Thaler the 2017 Economics Nobel prize is a good summary of the main motivation of this recent field of study : Behavioral Economics. The attempt to introduce other quantity than money to understand how humans share resources is not very new. Utility functions concept has yet been used by Bentham in the XVIIIth century and some other economists have taken into account that if compute rules most of the human world trades there is still yet, fortunately, a part of human brain decision that relies on psychological, chemical, conscious or unconscious but a certainly not negligible part of human behaviors that could not be computed, humanity could not be simulated just because rules which should be used to realize such a computation could not stand in a computer [5].

"We are not Econs" should probably mean at least this when Richard H.Thaler write it in "Nudge" [10]. So if we agree with him and we will try to understand economics this unmemoriable science, if Adam Smith [9] is a famous reference for the XXIth century traders we can not neglect that Egyptian scribes already compute resources, Cesar army obey to economics and mathematics rules and even early Australopithecus in caves already need to manage between their feelings, love, fear, anger, thirst... and a more objective quantity often finite that at this period could be the number of successfully hunted mammoths and so the quantity of food to share. This old principle should not be forgotten and maybe some economist who too much theorize complex financial product and who only try to maximize returns rate or their own total amount of money have lost this fundamental psychological rules formulated by Keynes [7] that says basically that we can only consume what we produce. Lavoisier also enounces a physical rule that leads to the same fact : "Rien ne se perd, rien ne se crée, tout se transforme". Our planet energy is finite so our food resources are also finite so trying to only maximize a bank account amount is certainly a stupid idea and understand how resources could be better shared taking into account as well as we can humans characteristics, feelings and preferences a better one.

How to share preferences ?

For sure, it implies to communicates, today the diversity of humans ways of communication as surely too complex to enable to us to pretend to treat this question in few pages so we will simplify the question using mathematical representations : we can not pretend to understand all mechanisms that rule humanity trades even not all mechanisms that rule only two individuals interactions we can only use approximation and compute on only mathematical object that can represent an incredible variety of more concrete reality manifestation. If we came "Back to Bentham" [3] with Chiappori & Al we can draw curves that are supposed to reflect not stupidly a piece of individual behavior : a Utility curve. If we agree with most behavioral economist such a utility curve should be concave. Globally a curve that starts from zero and finished to zero, grows, rich a maximum and decrease, so in a finite resource world excepted for a Kamikaze, a utility curve is concave. What does it mean ? It suggests that even if you will ever be happier to win much more money, the first 100 euros you win increase your happiness more than the next 100 euros and so on : the marginal utility of any kind of resource is a decreasing function so mathematically it means that second derivate of utility function is well concave. If we try to draw "happiness" in function of our "consumption" in an orthonormal repair, the curve we get is more a logarithm than an exponential. Wakker [11] have spent much time to study the shape of the Utility function ("Power Family"). If we can use a kind of utility function that is called CRRA for Constant Relative Risk Aversion and whose shape may be

$\mathbb{U}(r) = \sum w(p;\gamma)\frac{(1+r)^{1-\theta}-1}{1-\theta}(1+\lambda \mathbb{1}_{r<0})$ where r is a return rate, θ the risk aversion, p a

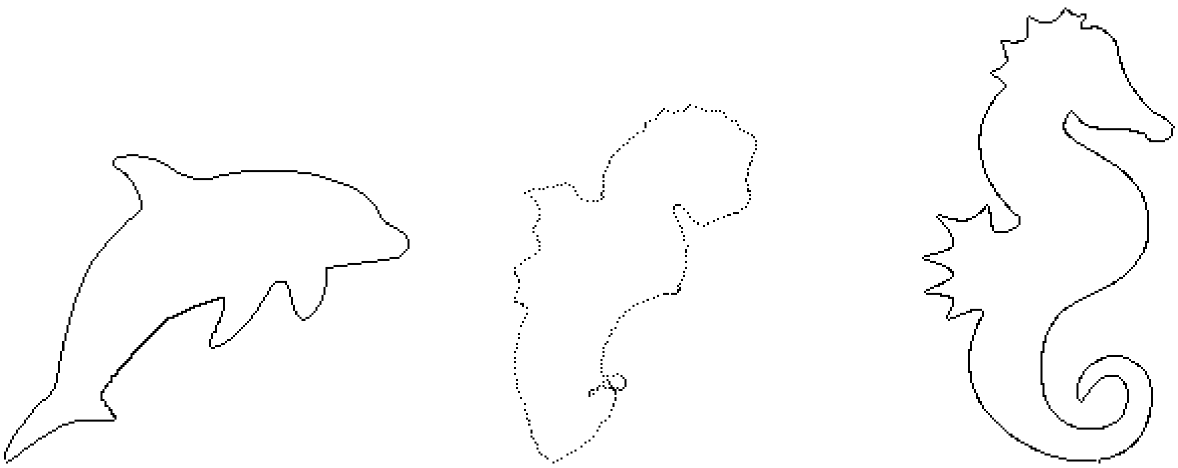


FIGURE 2 – Dolphin

probability of rate return, λ the loss aversion useless for positive rates and w a weight that could be given by the Prelec formula $w(p;\gamma) = \frac{p^\gamma}{(p^\gamma+(1-p)^\gamma)^{\frac{1}{\gamma}}}$, γ the probability deormation

We do not reject other modelization that may lead to another shape of utility function but we at least demand that those models respect the concavity of utility function. The first question that we will try to answer is :

"How could we aggregate two utility functions ?"

What does it mean ? In fact, a utility curve reflects not too bad a part of economic behavior of an individual we would like to be able to represent the behavior of a couple. We assume that we have succeeded to draw two curves that we can suppose to be concave, and we would like to draw a third one that could be the representation of the couple utility function : we have two entity that combined together form a third one entity, we have a profile for both each individual entities and we would like to get a curve that might be a not stupid representation of the entity resulting of the aggregation of the two individuals profiles : mathematically it is simple :

"How to aggregate two curves ?"

Historical context

- 1776 : Adam Smith "La richesse des nations"
  La main invisible : L'ensemble des actions individuelles des acteurs économiques guidés par définition uniquement par l'intérêt personnel de chacun contribuent à la richesse et au bien commun.
- 1921 : Frank H.Knight "Risk, Uncertainty and Profit"
- 1936 : John Meynard Keynes "Fundamentale Psychological Law"
- 1944 : Hayek "The road to serfdom"
- 1948 : Knight ""
  "Such a book as this may help to make a contribution to the modest but essential sector, the understanding of the mecanism of the open market as a method of cooperation. For this type of organisation must certainly have a huge role as long as men are men and neither bees in a hive nor pieces in a game where a few magnates struggles for power"
- 1953 : Maurice Allais (Nobel Prize & Allais' Paradox)
- 1974 : Hayek (Nobel prize)
- 1976 : Hayek (Money concurrency analyse) precursor concept of Bitcoins
- 1979 : Kahnman & Tversky (Prospect Theory)
- 1992 : Tversky & Kahnman (Cumulative Prospect Theory)
- 2001 : Chiappori, Fortin, Lacroix
- 2017 : Chiappori & Mazzocco

Ideological context

Between 1979 and 1992, Kahnman and Tversky critics the Expected Utility Theory and propose Prospect and Cumulative Prospect Theory rather than separable decision weight. The Prelec Formula help to understand Allais' paradox.

In 1988, Chiappori explain how household could be "collectively rational", in a couple, Pareto weight for utility sharing seems to explain equilibrium on Pareto frontier better than a Nash one, cooperation is proned.

In 2017, in "Static and Intertemporal Household Decision", Mazzocco explains which aspect of household decisions, different models can account for. Already describes in 2015 in "Household Survey" Static models : unitary, non-cooperative - Nash - or collective - Pareto - respectively consists in three optimisation problems :

- $max(u^{couple}(u_1, u_2))$
- $max(u_1)\&max(u_2))$
- $max(\lambda u_1 + (1 - \lambda)u_2))$

Intertemporal models are more complex, at least two differents state are used and ollows to modelise "before" and "after" an event and how a couple decision evolue before and after a wedding, a divorce or a working promotion. The experimentation tell us than a too strong shok might disturbe ore much than a slowly continuous variation.

FIGURE 3 – Egg

FIGURE 4 – Evolution

"How to aggregate two curves ?"

"Dolphin", "Egg", "Evolution" and "Tree" figures computed thanks to the python program in appendices give us artsy but rigorous hints to answer our main question : "How to aggregate two curves". We will see in seven steps how to formalised this intuition :

- State of the Art
- The joint Formula
- No need to be convex
- The limits
- Household economy consequences
- FED&BCE rates aggregation
- Experimentation

FIGURE 5 – Tree

State of the Art

The beginning of this work began in a behavioral economics context. Kahnman&Tversky [6] have set the basics of expected utility. Chiappori [1] and Mazzocco [2] have continued to examine how couples combined their utility curves but still, yet few works have been done on "how aggregate two curves". If we ask this question to google we only nd weird or irrelevant schemas.

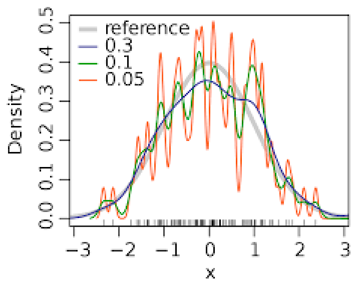


FIGURE 6 – Lissage Kernel (wikipédia)

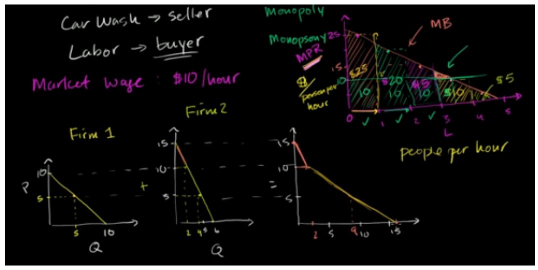


FIGURE 7 – Demand curves aggregation (kahnacademy)

In fact, this mathematical question that we need to solve to be able to understand couples the economy seems to have not been treated yet excepted maybe in signal theory, so stop reading and start thinking!

If we press the three first C on a piano, first we do not perceive the second. Next figure illustrates that : the first green oscillation is masked by firsts red and blue, so, in fact, the note in the middle sounds like if it was "inside" the upper and the downer ones.

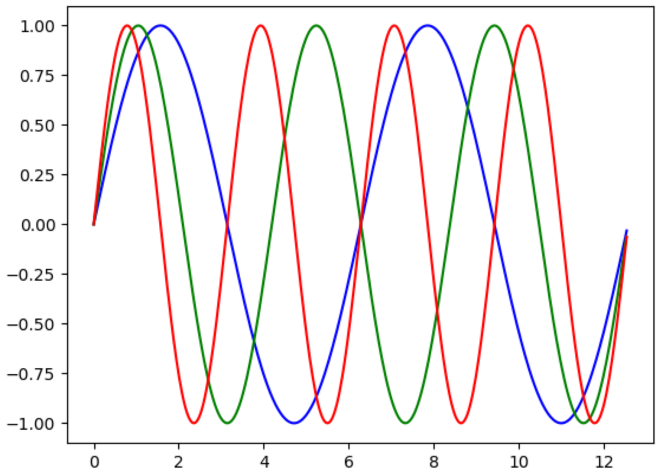


FIGURE 8 – sin(2x), sin(3x) and sin(4x)

If for each point $B[i] = (xB[i], yB[i])$ of the blue curve and $R[i] = (xR[i], yR[i])$ of the red curve we draw G the middle of the segment $[BR]$, coordinates of points G are $(\frac{xB+xR}{2}, \frac{yB+yR}{2})$.

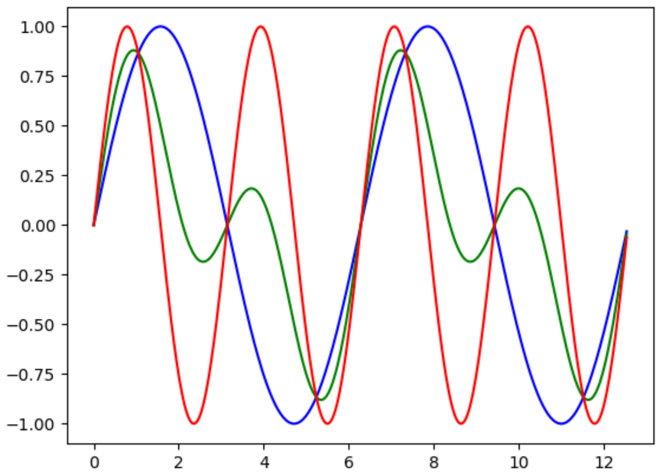


FIGURE 9 – Agregation of sin(2x) and sin(4x)

This green aggregated curve looks like to be a combination of sinus so it would be appreciable to be able to exprime the aggregated curve $h = f^{\&}g$ analyticaly from f and g.

The joint Formula

For each concave function, only one linear transformation leads to represent it in a unit-square in which the longer straight line join the min to the max of the function. After this transformation there is only one point which is the farthest from this line. If we consider two concave utility function [11] we have two points in the unit-square, therefore only one linear transformation of abscisses leads the middle abscissa of this segment to one half and the two abscisses extremity to $0 \leq p \leq \frac{1}{2} \leq 1-p \leq 1. \lambda$ from 0 to 1 allow to draw $h = f^{\&}g$ the aggregate curve according to the joint formula.

$$h(\frac{\lambda}{2}) = \frac{f(\lambda p)+g(\lambda(1-p))}{2}$$

$$h(1-\frac{\lambda}{2}) = \frac{f(1-\lambda(1-p))+g(1-\lambda p)}{2}$$

Do we really need to be convex ?

In fact we can apply joint formula to each piece of a curve so if we consider only convex partitioning of a curve which in fact is always an alternate of convex and concave part, if f is convex -f is concave and that wonderful remark allow us to apply the joint formula to every "reasonably regular" curve. By reasonably regular we hope to exclude demoniac pervers who would like to try to apply our python script on the $\mathbb{Q}$ indicatrice... a pixel coordinate is still an integer so please do not ask us to fulfill an impossible mission.

The limit of "Time reconstruction"

We can only aggregate curves. The tree example at the beginning of this paper has been enable through a long work which consists to draw the tree in an unique curve. In graph theory, the beautifull tree we have drawn is a degenerated tree and although it contain several thousands of pixels, a formal theorical description of the image is "a curved line between two point". So we could not use our python script to agregate a "x" with a "o". We have thorougly try to aggregate handwriting words, while we draw two lines whithout retiring our pen contact from the sheet we can use our python script to try to combine the two curve and reconstruct time to draw the curve which would have been drawn simutaneously if a third pen will have been moved exactly on the middle of both which have drawn the two curves. Trying to agregate "maths" and "economie" with $\frac{1}{2}$ and $\frac{2}{3}$ weight we get curious handwriting curve, "maths" seems to resist better to the transformation but may be the shortness of the word is a better explaination than a doubtfull mystical strength of the word concept...

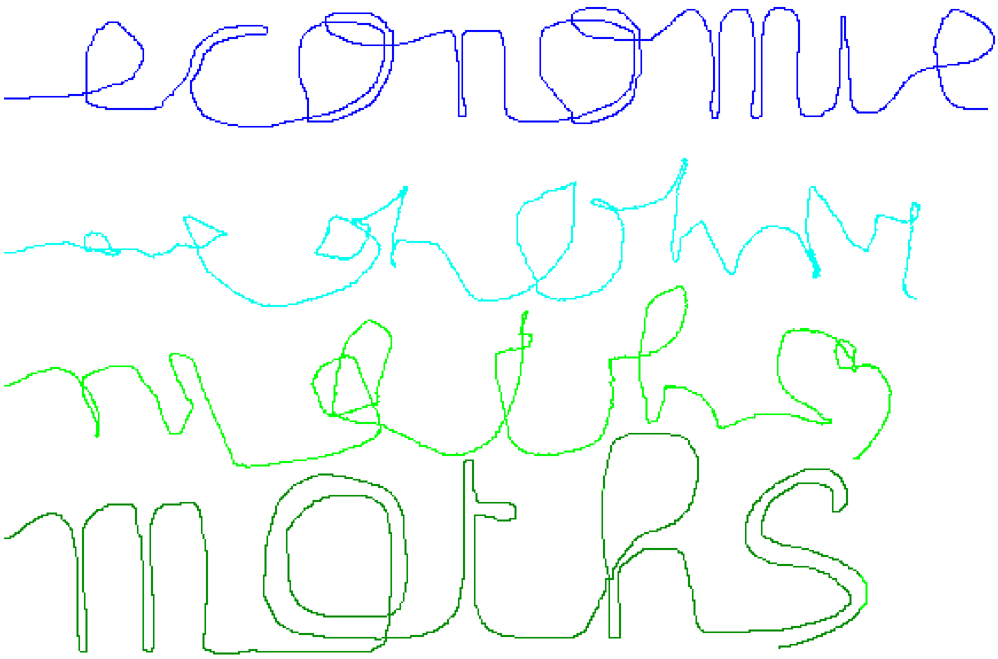


FIGURE 10 – maths $\frac{2}{3}, \frac{1}{3} - \frac{1}{3}, \frac{2}{3}$ economy

What about the economy ?

Risk profile aggregation has motive us, we do not still yet succed to demonstrate that couple utility profile could be computed with our python script but the curve we are able to draw from utility curves of man and woman appears to be a good candidate, validate this model could be the purpose of another paper. Figure11 shows utility curves profile for a risk aversion θ from 3 to 33 [4], according to our joint formula we expect an aggregated value of θ a little bit over the mean of both θ couple's members. But we also notice that couple utility aggregated curve depend of resources : The black line correspond to the utility of a kamikaze, we can suppose that he will save less money than his wife who is more risk averse so to fix our supposition we assume that the kamikaze in black save 20k, if his spouse save 33k, agregated curve between black and cyan ones will be near the yellow one and represent a θ near 18. In a less probable configuration where kamikaze and her spouse both succeed to save 33k the aggregated curve will be situated just under the red one (θ between 20 and 23). An other situation could be a kamikaze in black saving 30k and her spouse in cyan saving 20k and who share everything, their aggregated utility curve could be drawn between red and purple ones until the abscissa reaches 25k. On the other side if kamikaze save 17k and his wife 33k the couple utility curve will be more near the green one and ends at 20k. But if man and woman save both the same amount of resources the aggregated θ is just over the mean.

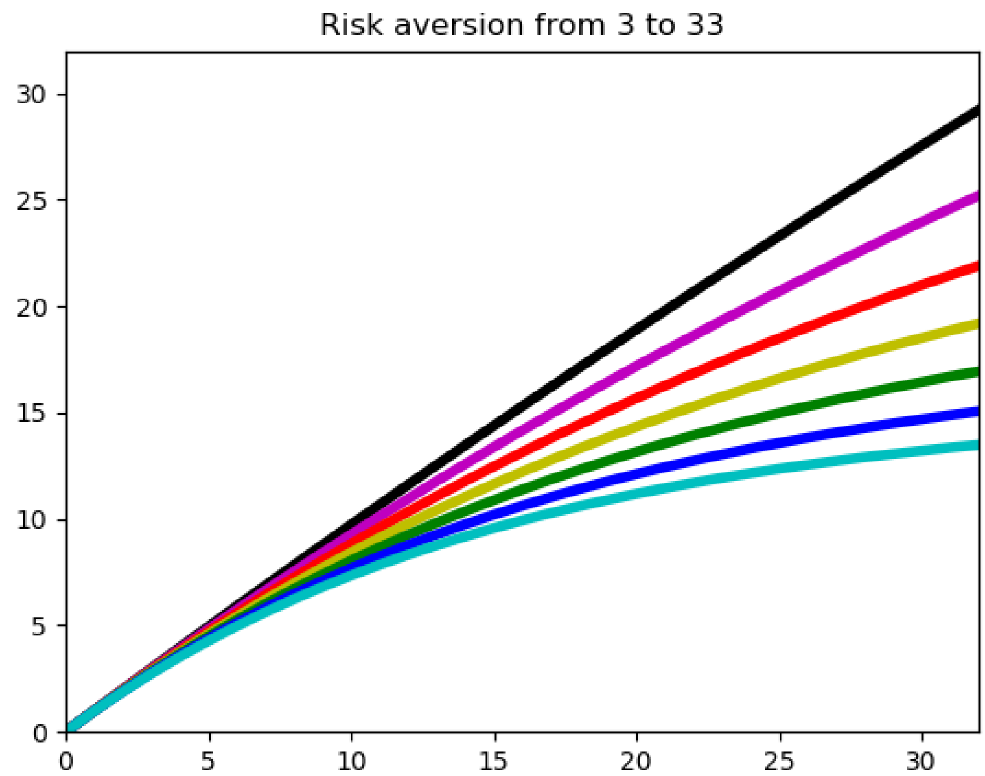
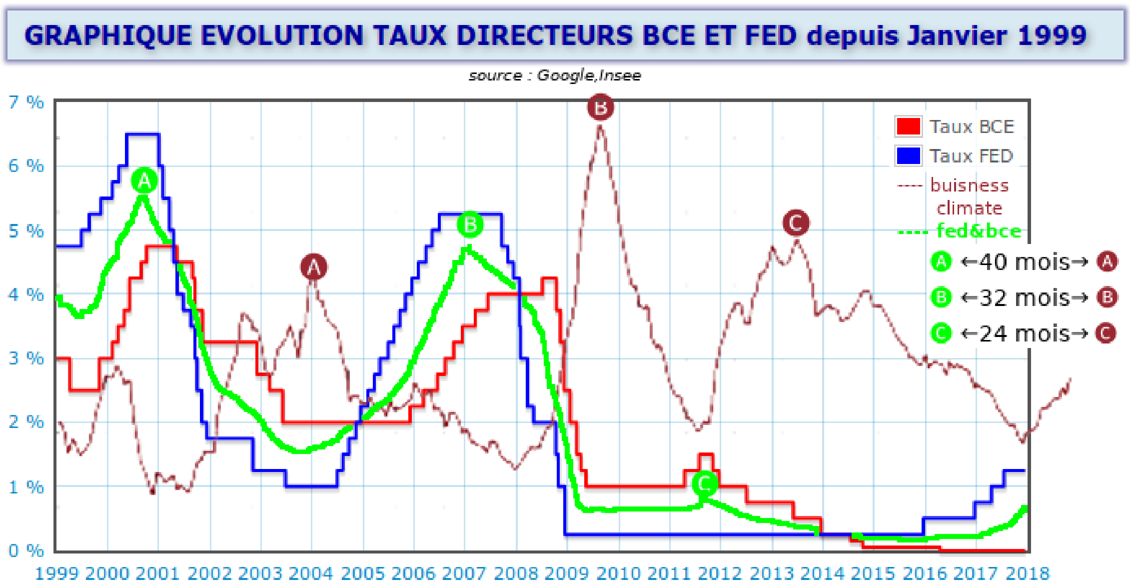


FIGURE 11 – $C_{\theta_{18}} \leq C_{\theta_3} \& C_{\theta_{33}} \leq C_{\theta_{23}}$

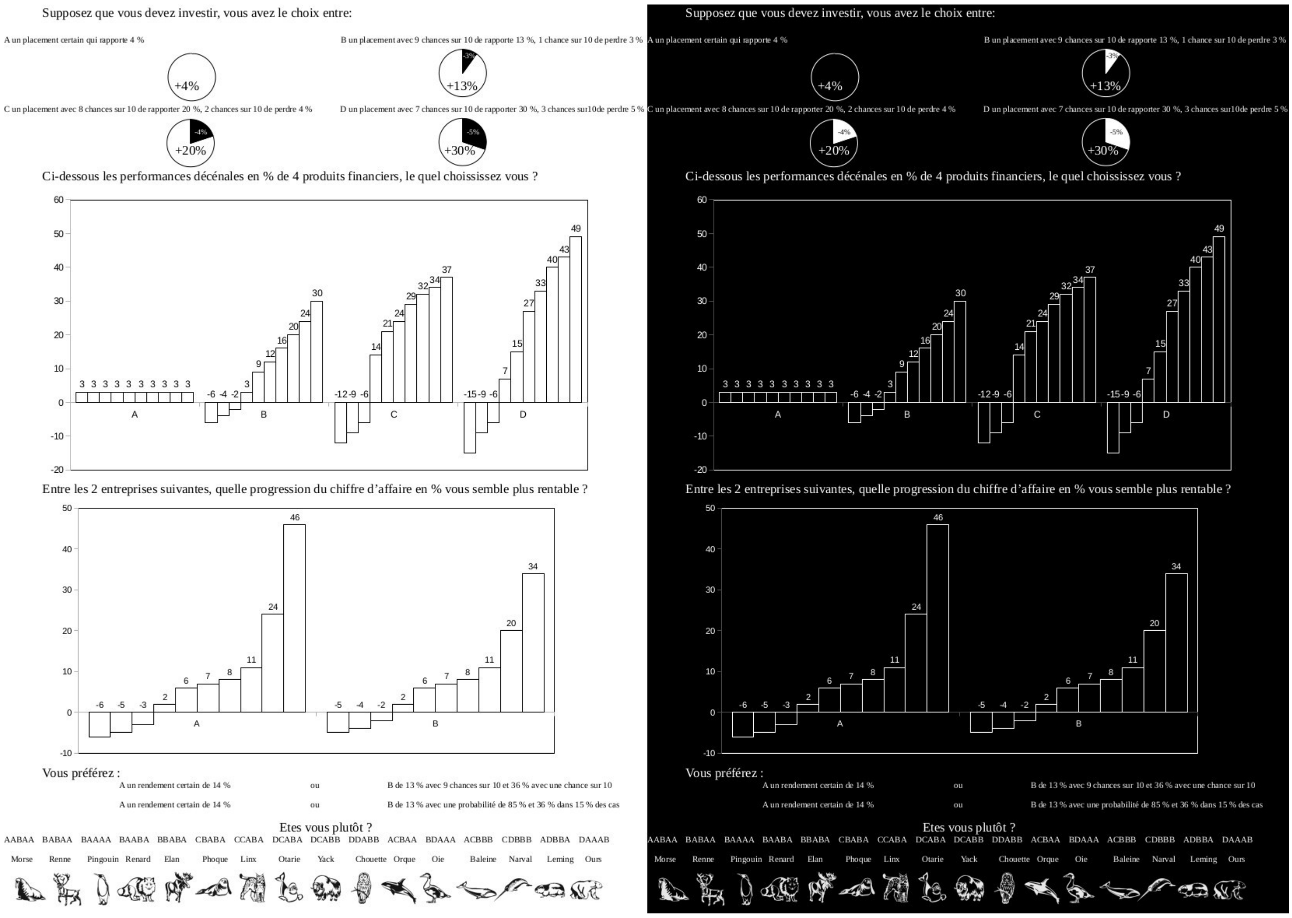
Concrete example : rates evolution, fed & bce

Centrales bank rates, FED (blue) and BCE (red) seems to draw a common world growth trajectory. It is in fact multiple representation of the same phenomenon with different time scale. If we agregate red and blue curves, we find a green curve which relatively correctly predict world growth with a lag which tends to reduce with world trades acceleration which is typical from our XXIth century beginning. The gap between centrales bank rates maxima A,B and C in green and main economic crisis A,B and C in red where almost 3 years at the begining of 2000's and a bit under 2 years actually. Actual trends seems to be a growing green curve so according to axes inversion, lower world growth is plausible.



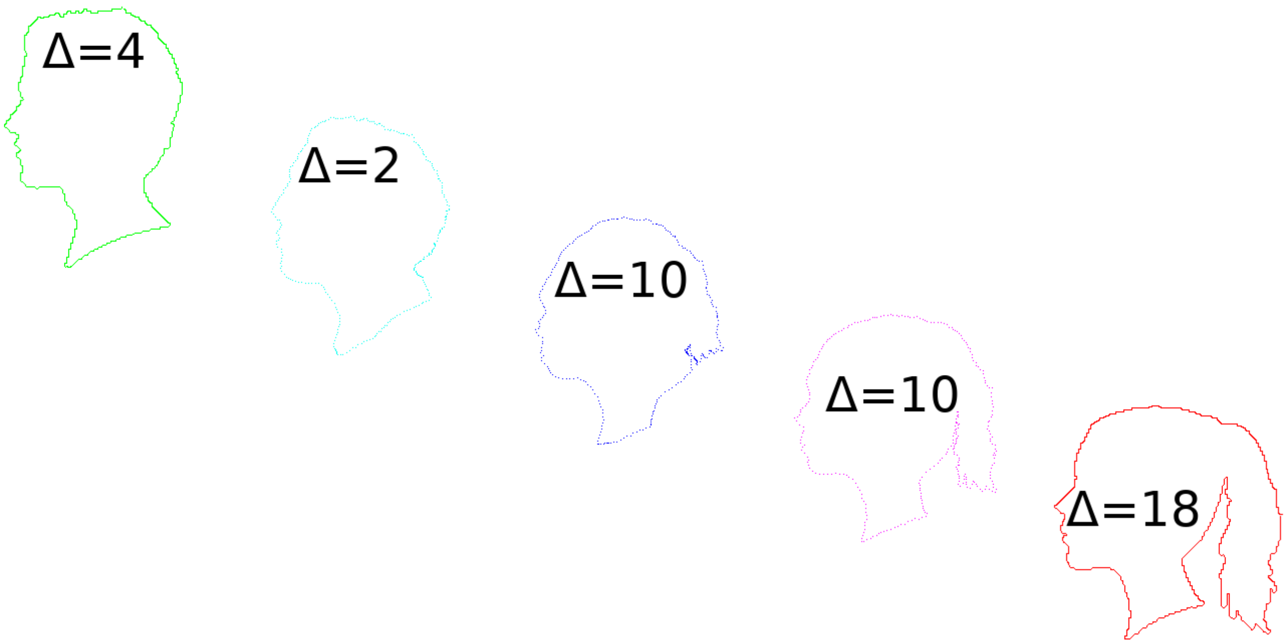
Experimentation

The following questionnary has been sent to 300 couples, yet only ten have answered, five single answers were also analysed. The couples profile answers have been classified into 7 risk level from 1 "very risky" to 7 "very cautious".



In the next figure we display the total number of level error between our model prediction and the reality of the answers : if we use man profile to predict the couple one we do 4 levels of errors while if we predict the couple profile only with the woman one we do 18 levels of error. A mix of man and woman profile predict better the couple profile, we compute our model for 3 value of our aggregation parameter : the value $\frac{1}{2}$ corresponding to the perfect symmetry in blue gives us 10 levels of error as much as the value $\frac{3}{4}$ wich consist in a centroïd curve between man and woman profile with 75% of weight for woman and 25% for man. We compute $\Delta = \sum_{i \in \{Couples\}} \delta_i$ where δ_i is the prediction error that is to say the difference between the risk level predict by the model and the risk level effectivly choosed by the couple.

So with the very few numbers of answers we currently have the better value for our aggregation coeficient seems to be $\frac{1}{4}$ that is to say that couple profile utility curve could in a first not so bad approximation modelized as the aggregation of man and woman profil according our Joint Formula by modifying the weight of each couple member and given 75% of weight to the man and 25% to the woman.



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```
import sys
import os

neighboor=[[-1,-1],[1,-1],[1,1],[-1,1],
           [0,-1],[1,0],[0,1],[-1,0]]

def file2bin(f, w, h):
  t = list(range(w*h))
  ff = open(f)
  l = ff.read()
  for i in range(w*h):
      o = l[122+3*(i)]
      if(o=='\x00'):
        t[i] = 1
      else:
        t[i] = 0
  ff.close()
  return t

def demizoom(t,w,h):
  tt = list(range((w/2)*(h/2)))
  for j in range(h/2):
    for i in range(w/2):
      tt[i+j*(w/2)] = 0
      for k in range(2):
        for l in range(2):
          if(i+k<w and j+l<h):
            tt[i+j*w/2]|=t[2*i+k+(2*j+l)*w]
  return tt

def zoom(t,w,h,z):
  tt = list(range(w*h*z*z))
  for j in range(h):
    for i in range(w):
      for k in range(z):
        for l in range(z):
          tt[z*i+k+(z*j+l)*z*w]=t[i+j*w]
  return tt

def dilate(i,w,h,nbcomp,compnum,t):
  iy = i / w
  ix = i-w*iy
  for k in range(8):
    jx = ix + neighboor[k][0]
    jy = iy + neighboor[k][1]
    if((jx>=0) and (jx<w) and (jy>=0) and (jy<h)):
      j = jx + jy * w
      if(t[j]!=0):
        if(compnum[j]<compnum[i]):
          compnum[j] = compnum[i]
          dilate(j,w,h,nbcomp,compnum,t)

#index of first node of next comp
def nextComp(i,w,h,t,compnum):
  while((i<w*h) and (compnum[i]>0 or
    (compnum[i]==0 and t[i]==0))):
    i+=1
  return i

def conexComp(t):
  nbcomp=0
  compnum = list(range(w*h))
  for i in range(w*h):
    compnum[i] = 0
  i = nextComp(0,w,h,t,compnum)
  while(i<w*h):
    nbcomp+=1
    compnum[i] = nbcomp
    dilate(i,w,h,nbcomp,compnum,t)
    i = nextComp(i,w,h,t,compnum)
  return [compnum,nbcomp]

def affiche(t,w,h):
  for j in range(h):
    line = ""
    for i in range(w):
      line += "%d" % t[i+w*j]
    print(line)

def numerote(t,w,h):
  s=0
  for j in range(h):
    for i in range(w):
      s = s+t[i+w*j]
  x = list(range(s))
  y = list(range(s))
  next = list(range(s))
  index = 0
  ix=0
  for iy in range(h):
    if t[ix+w*iy]>0:
      x[index]=ix
      y[index]=iy
      next[index]=0
      print("Debut Est")
  iy=0
  for ix in range(w):
    if t[ix+w*iy]>0:
      x[index]=ix
      y[index]=iy
      next[index]=0
      print(ix)
      print("Debut North")
  ix=w-1
  for iy in range(h):
    if t[ix+w*iy]>0:
```

```
      x[index]=ix
      y[index]=iy
      next[index]=0
      print("Debut West")
  iy=h-1
  for ix in range(w):
    if t[ix+w*iy]>0:
      x[index]=ix
      y[index]=iy
      next[index]=0
      print("Debut Sud")
  for index in range(s):
    ix = x[index]
    iy = y[index]
    for k in range(8):
      jx = ix + neighboor[k][0]
      jy = iy + neighboor[k][1]
      if((jx>=0) and (jx<w) and (jy>=0) and (jy<h)):
        j = jx + jy * w
        if(t[j]!=0 and(index<s-1)and
          ((index==0)or((jx!=x[index-1] or jy!=y[index-1]))
          and
          ((index==1)or(jx!=x[index-2] or jy!=y[index-2])))):
          x[index+1]=jx
          y[index+1]=jy
          next[index+1]=k
          if (k>3 and next[index]==k):
            break
  return [x,y,next,s]

def numeroteMod(x,y,next,s, mod):
  xx=list(range(s/mod))
  yy=list(range(s/mod))
  nextt=list(range(s/mod))
  for i in range(s/mod):
    xx[i]=x[i*mod]
    yy[i]=y[i*mod]
    nextt[i]=next[i*mod]
  return [xx,yy,nextt,s/mod]

def numeroteMul(x,y,next,s, mul):
  xx=list(range(s*mul))
  yy=list(range(s*mul))
  nextt=list(range(s*mul))
  for i in range(s):
    for j in range(mul):
      xx[i*mul+j]=x[i]
      yy[i*mul+j]=y[i]
      nextt[i*mul+j]=next[i]
  return [xx,yy,nextt,s*mul]

def drawComp(x,y,s,w,h):
  t=list(range(w*h))
```

```
  for j in range(h):
    for i in range(w):
      t[i+j*w] = 0
  print(len(t))
  print(s)
  for index in range(s):
    ix=x[index]
    iy=y[index]
    #print(ix)
    #print(iy)
    t[ix+iy*w] = 1
  return t

def extractComp(t,color,w,h):
  north = 0
  south = h-1
  est = w-1
  west = 0
  for i in range(w):
    for j in range(h):
      pix = i + w*j
      if(t[pix]==color):
        north = max(north,j)
        south = min(south,j)
        est = min(est,i)
        west = max(west,i)
  width = west-est
  height = north-south
  size = max(width,height)
  print(width, height)
  tt = list(range(size*size))
  for i in range(size):
    line = ""
    for j in range(size):
      tt[i+j*size] = 0
      if(i<width and j<height):
        pix = (south + j)*w + (est + i)
        if(t[pix]==color):
          line += 'x'
          tt[i+j*size] = 1
        else:
          line += ' '
          tt[i+j*size] = 0
  return [tt,size]

def comp2bmp(t,color,filename,w,h):
  north = 0
  south = h-1
  est = w-1
  west = 0
  for i in range(w):
    for j in range(h):
      pix = i + w*j
```

```
    if(t[pix]==color):
      north = max(north,j)
      south = min(south,j)
      est = min(est,i)
      west = max(west,i)
width = west-est+1
height = north-south+1
size = max(width,height)
print(width, height)
if(size<100):
  print("too small")
  return
else:
  if(size<4000):
    ff = open("4000x640.bmp")
    f = open(filename,"w")
  else:
    print("too large")
    return
l = ff.read()
buf = ""
for i in range(size):
  line = ""
  for j in range(size):
    if(i<width and j<height):
      pix = (south + j)*w + (est + i)
      if(t[pix]==color):
        line += 'x'
        buf += 'x'
      else:
        line += ' '
        buf += ' '
    else:
      line += '0'
      buf += '0'
for i in range(len(l)):
  if(i<122):
    f.write(l[i])
  else:
   y = (i - 122)/4000
   x = (i - 122) - 4000*y
   if(x<size and y<size):
     j = y + x * size
     if(buf[j]=='x'):
       f.write('\x00\x00\x00')
     else:
       f.write('\xff\xff\xff')
   else:
       f.write('\xff\xff\xff')
  f.close()

def mean(l,ll):
  lll=list(range(min(len(l),len(ll))))
  for i in range(min(len(l),len(ll))):
    lll[i]=(l[i]+ll[i])/2
  return lll

def ponder(l,ll,alpha):
  lll=list(range(min(len(l),len(ll))))
  for i in range(min(len(l),len(ll))):
    lll[i]=int(alpha*l[i]+(1-alpha)*ll[i])
  return lll

def pgcd(a,b,pgcd):
    c,d=max(a,b),min(a,b)
    while d>0:
        c,d=d,c%d
    pgcd = c
    return pgcd
if __name__ == '__main__':
  f = sys.argv[1]
  ff = sys.argv[2]
  w = int(sys.argv[3])
  h = int(sys.argv[4])
  t = file2bin(f,w,h)
  tt = file2bin(ff,w,h)
  [x,y,next,s] = numerote(t,w,h)
  [xx,yy,nextt,ss] = numerote(tt,w,h)
  m = pgcd(s,ss,1)
  M=s*ss/m

  [xM,yM,nextM,sM] = numeroteMul(x,y,next,s,20*s/m)
  [xxM,yyM,nexttM,ssM] = numeroteMul(xx,yy,nextt,ss,

  [X,Y,NEXT,S] = numeroteMod(xM,yM,nextM,sM,sM/5000)
  [XX,YY,NEXTT,SS] = numeroteMod(xxM,yyM,nexttM,ssM,

  ttt=drawComp(x+xx+ponder(
    X,XX,0.17)+ponder(X,XX,0.34)+ponder(
    X,XX,0.50)+ponder(X,XX,0.67)+ponder(X,XX,0.84),
    y+yy+ponder(
    Y,YY,0.17)+ponder(Y,YY,0.34)+ponder(
    Y,YY,0.50)+ponder(Y,YY,0.67)+ponder(Y,YY,0.84),
    s+ss+5*min(S,SS),w,h)
  filename = "agregate.bmp"
  comp2bmp(ttt,1,filename,w,h)
```

Animation

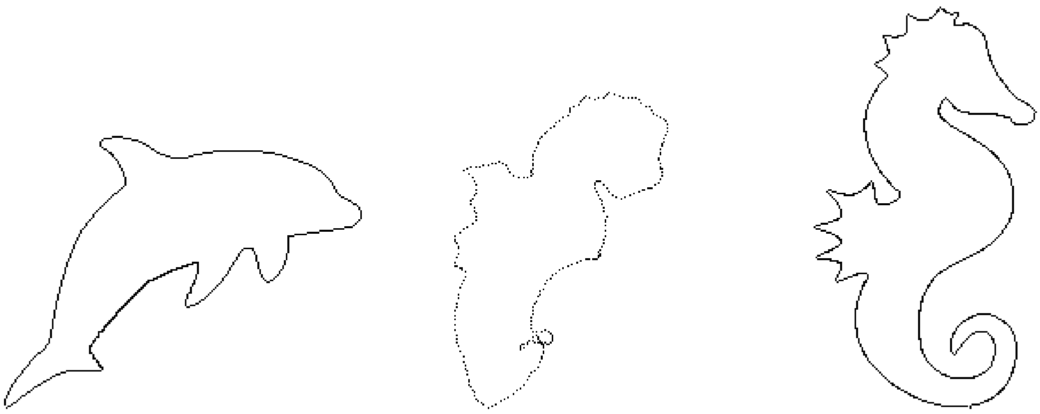

FIGURE 12 – Static python program result

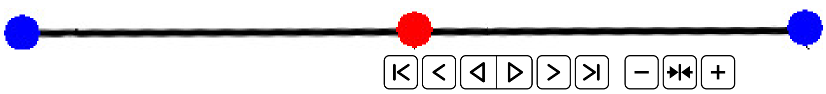

FIGURE 13 – Dynamic python program heuristic